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Scattering from the potential barrier $V = \cosh^{-2}\omega x$ from the path integration over $SO(1,2)$

H Ahmedov† and I H Duru†‡

† TUBITAK—Marmara Research Centre, Research Institute for Basic Sciences, Department of Physics, PO Box 21, 41470 Gebze, Turkey

‡ Trakya University, Mathematics Department, PO Box 126, Edirne, Turkey

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Abstract. Unitary irreducible representation of the group $SO(1,2)$ is obtained in the mixed basis, i.e. between the compact and non-compact bases, and new addition theorems are derived which are required in path integral applications involving a positively signed potential. The Green function for the potential barrier $V = \cosh^{-2}\omega x$ is evaluated from the path integration over the coset space $SO(1,2)/K$ where K is the compact subgroup. The transition and the reflection coefficients are given. Results for the moving barrier $V = \cosh^{-2}\omega(x - g_0 t)$ are also presented.

1. Introduction

Path integrations over group manifolds or over homogeneous spaces are frequently used for solving the path integrals of quantum mechanical potentials [1–3]. For example, Pöschl–Teller, Hulthen and Wood–Saxon potentials can be related to the group $SU(2)$ [4]. Path integrals over the $SU(1,1)$ manifold are studied for solving the modified Pöschl–Teller potential [5].

$V = -\cosh^{-2}\omega x$, i.e. the symmetric Rosen–Morse potential is the special case of potentials already mentioned in the above paragraph, which can also be solved by transforming its Green function into the Green function of the particle motion over the SO(3) manifold [6].

When the sign of the above potential is changed, that is when we consider the well known potential barrier $V = \cosh^{-2}\omega x$, which is related to the single soliton solution of the KdV equation, special attention is required. It is not the special case of the Rosen–Morse or the modified Pöschl–Teller potentials any longer. It is not related by coordinate transformation to them either. When one writes the e-value equation for the Laplace–Beltrami (LB) operator in the space of matrix elements of the unitary irreducile representations of $SO(1,2)$ realized in the compact basis, one arrives at the Schrödinger equation for the potential $V = \sinh^{-2}\omega x$ from which by the substitution $\omega x = \omega x' + \mathrm{i}\frac{\pi}{2}$ we come to the potential $V = -\cosh^{-2}\omega x'$. To obtain the positive sign before the latter potential one should diagonolize the LB operator in the space of matrix elements of the unitary irreducible representation constructed in the mixed basis, i.e. between the compact and non-compact bases. Such a necessity requries the derivation of a new addition theorem for these matrix elements to get the path integral solution. In fact the construction of the unitary ireducible representations in the mixed basis and the harmonic analysis on the double-sheeted hyperboloid in the hyperbolic coordinate

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system are the basic ingredients of the present note. The approach we adopt is of a general nature which can be used to obtain the path integral solutions in any homogenous space in any parametrization. It leads to the path integral solution for the new class of potentials (see section 2). The wavefunctions of these potentials correspond to the matrix elements of the unitary irreducible representations in the basis defined by the choice of the group decomposition.

In section 2 we briefly review the several possible decompositions of $SO(1,2)$ relevant to the coordinates employed in the coset spaces, which are double-sheeted and single-sheeted hyperboloids, and the cone.

In section 3 we formulate the LB operator for the coset space $SO(1,2)/SO(2)$. Following the derivation of $SO(1,2)$ matrix elements in the mixed basis, we diagonolize the LB operator, and then arrive at the Schrödinger equation of the potential barrier $V = \cosh^{-2}\omega x$. Normalized wavefunctions and spectrum are given.

In section 4 we present the path integral formulation over the homogeneous space $SO(1,2)/SO(2)$. Starting from the short-time-interval Kernel and making use of the newly derived addition theorems we expand the short-time-interval Kernel in terms of the group matrix elements.

In section 5, we study the path integration for the potential barrier $V = \cosh^{-2}\omega x$. Transmittion and reflection coefficients are given. Formulae for the barrier moving with a constant speed g_0, i.e. for $V = \cosh^{-2}\omega(x - g_0 t)$, which is more relavent to the solitonic potential, are also presented.

2. Decompositions of the group $SO(1,2)$ and related quantum systems

To express the group $SO(1,2)$ in the decomposed forms the following one-parameter subgroups can be employed:

$$a = \begin{pmatrix} \cosh\alpha & 0 & \sinh\alpha \\ 0 & 1 & 0 \\ \sinh\alpha & 0 & \cosh\alpha \end{pmatrix} h = \begin{pmatrix} \cosh\beta & \sinh\beta & 0 \\ \sinh\beta & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} n = \begin{pmatrix} 1 + \frac{t^2}{2} & t & \frac{t^2}{2} \\ t & 1 & t \\ -\frac{t^2}{2} & -t & 1 - \frac{t^2}{2} \end{pmatrix} \tag{1}$$

where

$$\alpha, \beta \in (-\infty, \infty) \qquad \psi \in (0, 2\pi) \qquad t \in (-\infty, \infty). \tag{2}$$

$G = SO(1,2)$ leaves the form $(x, x) = x_0^2 - x_1^2 - x_2^2$ invariant. There are three possibilities:

2.1. Double-sheeted hyperboloid $M : (x, x) > 0$ [7,8]

The vector $\dot{\xi} = (1, 0, 0) \in M$ is the stationary point of the compact subgroup k as $\dot{\xi} k = \dot{\xi}$. Thus the decompositions of the group SO(1,2) related to the double-sheeted hyperboloid M is

$$g = kb \qquad g \in SO(1,2) \qquad k \in SO(2). \tag{3}$$

The choice of the boost b defines the coordinate systems on M:

(i) $b = ak$ or the Cartan decomposition of the group $g = kak'$ defines the spherical coordinate parametrization of M. This choice is convenient for studying the quantum mechanical potential $1/\sinh^2\alpha$.

(ii) $b = ah$ or the non-compact Cartan decomposition of the group $g = kah$ defines the hyperbolic coordinates on M. It is suitable to the quantum mechanical system with potential $1/\cosh^2\alpha$. This case is the subject of the present work.

(iii) $b = an$ or the Iwasawa decomposition of the group $g = kan$ defines the parabolic coordinates on M which leads to the quantum mechanical system with potential $V = \exp(\alpha)$.

2.2. Single-sheeted hyperboloid $\overline{M} : (x, x) < 0$ [7,8]

The vector $\dot{z} = (0, 0, 1) \in \overline{M}$ is the stationary point of the non-compact subgroup h as $\dot{z}h = \dot{z}$. The decompositions of the group $SO(1,2)$ related to the single-sheeted hyperboloid $\overline{M}$ has the form

$$g = hb \qquad g \in SO(1,2) \qquad h \in SO(1,1) \tag{4}$$

with the possible choices of the boost b given as the following:

(i) $b = ak$ or the non-compact Cartan decomposition of the group $g = hak$ defines the spherical coordinate parametrization of $\overline{M}$ which produces the potential $-1/\cosh^2\alpha$.

(ii) $b = (aI^{\varepsilon}h, kI^{\varepsilon}h)$ or decomposition of the group $g = (haI^{\varepsilon}h, hkI^{\varepsilon}h)$ [9] defines the hyperbolic coordinates on $\overline{M}$. Here I is the metric tensor given by

$$I = \mathrm{diag}(1, -1, -1) \tag{5}$$

and $\varepsilon = 0, 1$. This decomposition is suitable to the quantum mechanical system with potentials $-1/\sinh^2\alpha$ and $-1/\sin^2\phi$.

(iii) $b = aI^{\varepsilon}n$ leads to the non-compact Iwasawa decomposition of the group $g = kaI^{\varepsilon}n$ and defines the parabolic coordinates on $\overline{M}$. The related quantum mecanical system is $V = -\exp(\alpha)$.

2.3. Cone $M_0 : (x, x) = 0$

The vector $\dot{y} = (1, 0, 1) \in M_0$ is the stationary point of the nilpotent subgroup n as $\dot{y}n = \dot{y}$. The decompositions of the group $SO(1,2)$ related to the cone M_0 has the form $g = nb$ with b having the following forms:

(i) $b = ak$ is the Iwasawa decomposition which defines spherical coordinates on M_0.

(ii) $b = aI^{\varepsilon}$h is the non-compact Iwasawa decomposition which defines hyperbolic coordinates on M_0.

(iii) $b = an^T$ is the Gauss decomposition which defines spherical coordinates on M_0.

It is impossible to relate quantum systems with the cone because the metric tensor of M_0 is degenerate. This space is used for the construction of the irreducible representations [10]. To construct the irreducible representations in the mixed basis we simultaneously have to use the realizations given by (i) and (ii) (see appendix A).

3. The double-sheeted hyperboloid in hyperbolic coordinates

We decompose the group $G = SO(1,2)$ as

$$g = hak. \tag{6}$$

Starting from the stationary point $\dot{\xi} = (1, 0, 0)$ we cover all the homogeneous space M by the act of the group elements as $x = \dot{\xi}g$. Using (1) we get the parametrization of M:

$$\xi = \dot{\xi}g = \dot{\xi}ah = (\cosh\alpha\cosh\beta, \cosh\alpha\sinh\beta, \sinh\alpha). \tag{7}$$

The metric tensor and the Laplace–Beltrami operator [11] of M are

$$g_M = \mathrm{diag}(-1, -\cosh^2\alpha) \qquad \det g_M = \cosh^2\alpha \tag{8}$$

and

$$\Delta = -\partial_\alpha^2 - \tanh\alpha\,\partial_\alpha - \cosh^{-2}\alpha\,\partial_\beta^2. \tag{9}$$

We will write down the e-value equation for the above LB operator in the space of $SO(1,2)$ matrix elements which are evaluated between the compact and the non-compact basis. The matrix elements of the unitary principle series of the group $SO(1,2)$ in the mixed basis (see appendix A) given by

$$d^\sigma_{\mu k}(g) = \langle\mu|T^\sigma(g)|k\rangle \tag{10}$$

are the e-functions of the invariant differentiale operator Δ:

$$\Delta d^\sigma_{\mu k}(g) = -\sigma(1+\sigma)d^\sigma_{\mu k}(g). \tag{11}$$

Here σ is the weight of the representation

$$\sigma = -\tfrac{1}{2} + \mathrm{i}\rho \qquad \rho\in(0,\infty) \tag{12}$$

and $|k\rangle$ and $|\mu\rangle$ are the compact and non-compact bases corresponding to the the degrees of freedom ϕ and β respectively.

Since we are dealing with the coset space $M = G/K$ with $K = SO(2)$, we do not need the full set of the matrix elements (10); instead we employ

$$d^\sigma_{\mu 0_k}(ha) = \langle\mu|T^\sigma(ha)|0_k\rangle. \tag{13}$$

Writing $d^\sigma_{\mu 0_k}(ha)$ as

$$d^\sigma_{\mu 0}(ha) = \exp(\mathrm{i}\mu\beta)d^\sigma_{\mu 0}(a) = \exp(\mathrm{i}\mu\beta)(\det g_M)^{-1/4}\Psi^\sigma_\mu(\alpha) \tag{14}$$

the e-value equation (12) becomes

$$(-\partial_\alpha^2 + (\mu^2+\tfrac{1}{4})\cosh^{-2}\alpha + \tfrac{1}{4})\Psi^\sigma_\mu(\alpha) = -\sigma(\sigma+1)\Psi^\sigma_\mu(\alpha) \tag{15}$$

which is equivalent to the Schrödinger equation for the potential barrier $V = \cosh^{-2}\alpha$ with an extra constant energy shift of $\frac{1}{4}$. Note that if we were to diagonolize the operator (9) in the space of matrix elements written between purely compact basis, the sign of potential would be negative. The wavefunctions of the Schödinger equation (15) are given in terms of the Legendre functions [12] by

$$\Psi^\sigma_\mu(\alpha) = \frac{\cosh^{1/2}(\pi\rho)\cosh^{-1/2}(\alpha)}{\cosh(\pi\rho) - \mathrm{i}\sinh(\pi\mu)}(P^{i\mu}_\sigma(\mathrm{i}\sinh\alpha) + P^{i\mu}_\sigma(-\mathrm{i}\sinh\alpha)) \tag{16}$$

which are normalized as (see appendix B)

$$\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\Psi^\sigma_\mu\overline{\Psi^{\sigma'}_{\mu'}} = \frac{\delta(\mu-\mu')\delta(\sigma-\sigma')}{\rho\tanh\pi\rho}. \tag{17}$$

4. Path integration over the coset space $M = G/K$

The probability amplitude for the particle of 'moment of inertia' m, to travel in the space M from the point ξ_a to ξ_b in the time interval T is expressed by the path integral which in the time-graded formulation is given by

$$K(\xi_a, \xi_b; T) = \lim_{n\to\infty} \int \prod_{j=1}^{n} \mathrm{d}\xi_j \prod_{j=1}^{n+1} K(\xi_{j-1}, \xi_j; \varepsilon) \tag{18}$$

with $T = (n+1)\varepsilon$ and $\mathrm{d}\xi_j = (\det g_M)^{1/2}\, \mathrm{d}\alpha_j\, \mathrm{d}\beta_j$.

The Kernel connecting the points ξ_{j-1} and ξ_j separated by the small time interval $t_j - t_{j-1} = \varepsilon$ is [4]

$$K_j = K(\xi_{j-1}, \xi_j; \varepsilon) = \left(\frac{m}{2\mathrm{i}\pi\varepsilon}\right)^{3/2} \exp(\mathrm{i}S_j) \tag{19}$$

where S_j is the short-time-interval action

$$S_j = \frac{m}{2\varepsilon}\delta^2_{j-1,j}. \tag{20}$$

The invariant distance between the points is

$$\delta^2_{j,j-1} = (\xi_j - \xi_{j-1}, \xi_j - \xi_{j-1}) = 2 - 2\cosh\theta_{j-1,j} \tag{21}$$

with

$$\cosh\theta_{j-1,j} = \cosh\alpha_{j-1}\cosh\alpha_j\cosh(\beta_{j-1} - \beta_j) - \sinh\alpha_{j-1}\sinh\alpha_j. \tag{22}$$

The short-time-interval Kernel (19) can be expanded in terms of the Legendre functions $P_\sigma(\cosh\theta)$ as (with $\sigma = -\frac{1}{2} + \mathrm{i}\rho$)

$$K_{j-1,j} = \int_0^\infty \mathrm{d}\rho\, \rho \tanh(\pi\rho) C_\sigma P_\sigma(\cosh\theta_{j-1,j}). \tag{23}$$

From the orthogonality condition

$$\int_1^\infty \mathrm{d}z\, P_\sigma(z)\overline{P_{\sigma'}(z)} = \frac{1}{\rho\tanh\pi\rho}\delta(\sigma - \sigma') \tag{24}$$

the coefficients C_σ are obtained:

$$C_\sigma = -\frac{m}{\sqrt{\pi\varepsilon}} \exp\left(-\frac{\mathrm{i}\pi}{\varepsilon}\right) K_{i\rho}\left(-\frac{m}{\varepsilon}\right). \tag{25}$$

Here $K_{i\rho}$ is the MacDonald function. In $\varepsilon \to 0$ limit, by using the asymptotic form of the MacDonald function [13] we can write the short-time-interval Kernel (19) as

$$K_j \simeq \int_0^\infty \mathrm{d}\rho\, \rho \tanh\pi\rho \exp\left(\frac{\mathrm{i}\varepsilon}{2m}\sigma(\sigma+1)\right) P_\sigma(\cosh\theta_j). \tag{26}$$

With the help of the addition theorem (see appendix C) for the complete set of functions on the homogeneous space M we get

$$K_j = \int_{-\infty}^{\infty} \mathrm{d}\mu \int_0^\infty \mathrm{d}\rho\, \rho\tanh\pi\rho \exp\left(\frac{\mathrm{i}\varepsilon}{2m}\sigma(\sigma+1)\right) d^\sigma_{\mu 0}(\xi_{j-1})\overline{d^\sigma_{\mu 0}(\xi_j)}. \tag{27}$$

We first insert the above form of the short-time-interval Kernel into (18); then by making use of the orthogonality condition (17) we can execute the $\prod_{j-1}^{n} \mathrm{d}\xi$ integrals:

$$K(\xi_a, \xi_b; T) = \int_{-\infty}^{\infty} \mathrm{d}\mu \int_0^\infty \mathrm{d}\rho\, \rho\tanh\pi\rho \exp\left(-\frac{\mathrm{i}(\rho^2 + \frac{1}{4})}{2m}T\right) d^\sigma_{\mu 0}(\xi_a)\overline{d^\sigma_{\mu 0}(\xi_b)}. \tag{28}$$

By using the addition theorem the Kernel (28) can be written in the form

$$K(\xi_a, \xi_b; T) = \int_0^\infty \mathrm{d}\rho\, \rho \tanh \pi\rho \exp\left(-\mathrm{i}\frac{\rho^2 + \frac{1}{4}}{2m}T\right) P_\sigma(\cosh\theta_{ab}) \tag{29}$$

where $\cosh\theta_{ab}$ depends on the coordinates of the points a and b through the relation defined by (22).

The Fourier transform of (29) can be calculated to obtain the energy-dependent Green function $G(\xi_a, \xi_b; E)$:

$$G(\xi_a, \xi_b; E) = \int_0^\infty \exp(\mathrm{i}ET) K(\xi_a, \xi_b; T) = 2m Q_{-\frac{1}{2}-\mathrm{i}\sqrt{(2mE-\frac{1}{4})}}(\cosh\theta_{ab}) \tag{30}$$

where Q is the Legendre function of the second kind. In deriving (30) we used the connection between the Legendre functions of the first and second kind [14]

$$\int_0^\infty \mathrm{d}x\, \frac{x\tanh\pi x}{a^2 + x^2} P_{-\frac{1}{2}+\mathrm{i}x}(z) = Q_{a-\frac{1}{2}}(z). \tag{31}$$

5. Path integral for the potential barrier $V = V_0 \cosh^{-2}(\omega x)$

The phase space path integral for the particle of mass m moving under the influence of the potential barrier $V = V_0\cosh^{-2}(\omega x)$ is

$$K(x_a, x_b; T) = \int \mathrm{D}x\, \mathrm{D}p_x\, \exp\left(\mathrm{i}\int_0^T \mathrm{d}t\, (px - p_x^2/2m - V_0\cosh^{-2}\omega x)\right) \tag{32}$$

which is in the time-graded formulation equal to

$$\begin{aligned} K(x_a, x_b; T) = \lim_{n\to\infty} \int \prod_{j=1}^{n} \mathrm{d}x_j \prod_{j=1}^{n+1} \mathrm{d}\frac{p_{xj}}{2\pi} \prod_{j=1}^{n+1} \exp[\mathrm{i}(p_{xj}(x_j - x_{j-1}) \\ -\varepsilon p_{xj}^2/2m - \varepsilon V_0 \cosh^{-2}\omega x_j)]. \end{aligned} \tag{33}$$

The phase space formulation (32) easily enables us to establish the connection between our quantum mechanical problem and the path integration over the coset space $M = SO(1,2)/K$. In fact when we consider the Hamiltonian H_M for the particle motion over the coset space $M = G/K$ (recall the Schödinger equation (15))

$$H_M - \frac{1}{4} \Rightarrow \frac{1}{2m}(p_\alpha^2 + p_\beta^2\cosh^{-2}\alpha) - \frac{1}{4}\left(\frac{\omega^2}{2m}\right) \tag{34}$$

we observe that the Hamiltonian in the action of (32) resembles the above Hamiltonian with the momentum p_β fixed to the value $p_\beta = \sqrt{(2mV_0)}$. In writing (34) we introduced the corrections due to parameters ω and $(2m)$ which were equal to 1 in sections 3 and 4. Thus we can convert the path integral (32) into the path integral for the motion in the space $M = G/K$. We first rescale x by $\omega x = \alpha$; $p_x = \omega p_\alpha$ and arrive at

$$K(x_a, x_b; T) = \omega \int \mathrm{D}\alpha\, \mathrm{D}p_\alpha\, \exp[\mathrm{i}\int_0^{\omega^2 T} \mathrm{d}t\, (p_\alpha\alpha - p_\alpha^2/2m - V_0\cosh^{-2}\alpha)]. \tag{35}$$

We then rewrite the potential term in the above path integral by introducing an auxiliary dynamics by extending the phase space with the identity

$$\begin{aligned} \exp\left[-\mathrm{i}\left(\int_0^{\omega^2 T} \mathrm{d}t\, V_0\cosh^{-2}\alpha\right)\right] = \int \mathrm{d}\beta_b\, \exp(-\sqrt{(2mV_0/\omega^2)}(\beta_b - \beta_a)) \\ \times \lim_{n\to\infty} \int \prod_{j=1}^{n} \mathrm{d}\beta_j \prod_{j=1}^{n+1} \mathrm{d}\frac{p_{\beta j}}{2\pi} \prod_{j=1}^{n+1} \exp\left(\mathrm{i}(p_{\beta j}(\beta_j - \beta_{j-1}) - \frac{\omega^2\varepsilon p_{\beta_j}^2}{2m\cosh^2\alpha}\right) \end{aligned} \tag{36}$$

which can be proved by direct calculation. Note that the phase space formulation is essential for the above identity which establishes the connection between our quantum mechanical problem and the partical mation over M. The identity of (36) enables us to re-express (35) as

$$K(x_a, x_b; T) = \omega \int \mathrm{d}\beta_b \, \exp(-\sqrt{(2mV_0/\omega^2)}(\beta_b - \beta_a)) \exp\left(\mathrm{i}\frac{\omega^2}{8m}T\right) K_M(\xi_a, \xi_b; T) \tag{37}$$

where K_M is the Kernel for the motion over the manifold $M = G/K$ which is studied in the previous chapter. The factor $\exp(\mathrm{i}\frac{\omega^2}{8m}T)$ in the above equation reflects (see equations (15) and (34)) the $\frac{1}{4}$ energy difference between the potential barrier $\cosh^{-2}\omega x$ and the particle motion over the coset space M. We then insert the expression (28) into (37), use (14) for the matrix elements $d^{\sigma}_{\mu 0}$ and arrive at

$$\begin{aligned} K(x_a, x_b; T) = 2\pi\omega(\cosh\omega x_a \cosh\omega x_b)^{-1/2} \int \mathrm{d}\rho \, \rho \tanh\pi\rho \exp\left(-\mathrm{i}\frac{\rho^2\omega^2}{2m}T\right) \\ \times \psi^{\sigma}_{\frac{(2mV_0)^{1/2}}{\omega}}(\omega x_a)\overline{\psi^{\sigma}_{\frac{(2mV_0)^{1/2}}{\omega}}(\omega x_b)} \end{aligned} \tag{38}$$

which displays the wavefunctions. The asymptotic form of the wavefunctions are

$$\lim_{x\to\infty} \psi^{\sigma}_{\frac{(2mV_0)^{1/2}}{\omega}}(\omega x) \simeq \frac{\Gamma(\frac{1}{2} - \mathrm{i}\rho)}{\Gamma(\frac{1}{2} - \mathrm{i}(\rho + \frac{(2mV_0)^{1/2}}{\omega}))} \exp(-\mathrm{i}\rho\omega x) \tag{39}$$

$$\lim_{x\to-\infty} \psi^{\sigma}_{\frac{(2mV_0)^{1/2}}{\omega}}(\omega x) \simeq \frac{\Gamma(\frac{1}{2} - \mathrm{i}\rho)}{\Gamma(\frac{1}{2} - \mathrm{i}(\rho + \frac{(2mV_0)^{1/2}}{\omega}))} (T\exp(-\mathrm{i}\rho\omega x) + R\exp(\mathrm{i}\rho\omega x)) \tag{40}$$

in which the transition and the reflection coefficients are identified as

$$T = \frac{\Gamma(\frac{1}{2} + \mathrm{i}(\rho + \frac{(2mV_0)^{1/2}}{\omega}))\Gamma(\frac{1}{2} + \mathrm{i}(\rho - \frac{(2mV_0)^{1/2}}{\omega}))}{\Gamma(\mathrm{i}\rho)\Gamma(1 + \mathrm{i}\rho)} \tag{41}$$

$$R = \frac{\Gamma(\frac{1}{2} + \mathrm{i}(\rho + \frac{(2mV_0)^{1/2}}{\omega}))\Gamma(\frac{1}{2} + \mathrm{i}(\rho - \frac{(2mV_0)^{1/2}}{\omega}))\Gamma(-\mathrm{i}\rho)}{\Gamma(\frac{1}{2} + \mathrm{i}\frac{(2mV_0)^{1/2}}{\omega})\Gamma(\frac{1}{2} - \mathrm{i}\frac{(2mV_0)^{1/2}}{\omega})\Gamma(\mathrm{i}\rho)}. \tag{42}$$

If one considers the same potential barrier in motion with a constant speed

$$V(x, t) = \frac{V_0}{\cosh^2\omega(x - g_0 t)} \qquad g_0 = \text{constant} \tag{43}$$

the Kernel becomes [15]

$$K_{g_0}(x_a, x_b; T) = \exp\left(-\mathrm{i}\frac{m}{2}g_0^2 T\right) \exp(-\mathrm{i}mg_0(x_b - x_a)) K(x_a - g_0 t_a, x_b - g_0 t_b : T). \tag{44}$$

Here the form of K is given by (38). From the above formula the wavefunctions are recognized as

$$\psi_{g_0}(x, t) = \exp\left(-\mathrm{i}\frac{m}{2}g_0^2 t\right) \exp(-\mathrm{i}mg_0 x) \psi^{\sigma}_{\frac{(2mV_0)^{1/2}}{\omega}}(\omega(x - g_0 t)) \tag{45}$$

where $\psi(\omega(x - g_0 t))$ is obtained from the static one given in (16) and (38) by simply replacing x by $x - g_0 t$. For finite values of time variable t the transition and the reflection coefficients remain in the static forms of (41) and (42).

Inspecting the limiting forms of T and R we obtains

$$\left|\frac{T}{R}\right| \to \infty \text{ as } \rho \to \infty \qquad \text{and} \qquad \left|\frac{T}{R}\right| \to 0 \text{ as } \rho \to 0. \tag{46}$$

We see that the low-energy waves are mostly reflected, while the high-energy waves are more easily transmitted through the barrier.

Inspecting (47) and the asymptotic forms as $x \to \infty$ we observe that the barrier motion contributes the following constant additional term to the energy:

$$\Delta E = \frac{mg_0^2}{2} - \rho\omega g_0. \tag{47}$$

The first term of the above extra energy is of the kinetic energy type (as g_0 has the dimension of velocity). It is also interesting that the barrier motion introduces the extra undulation of a Doppler nature through the $\exp(-\mathrm{i}mg_0x)$ term in the wavefunction.

Appendix A. The unitary irreducible representations of the group $SO(1, 2)$ in the mixed basis

We will construct the irreducible representations of the pseudo-orthogonal group $G = SO(1, 2)$ in the space of the infinitely differentiable homogeneous functions $F(y)$ with the homogenity degree σ on the cone Y: $[y, y] = 0$.

$$T^\sigma(g)F(y) = F(yg) \qquad g \in G, \ \ y \in Y \tag{A.1}$$

$$F(ay) = a^\sigma F(y) \qquad a \in R, \ \ \sigma \in C. \tag{A.2}$$

In order to construct the matrix elements of the representation in the mixed basis we have to define the cone in two coordinate systems corresponding to these bases. Use the Iwasawa decompositions for the group $SO(1, 2)$ [7]:

$$g = n(t)a(\theta)k(\phi) \qquad g \in G \tag{A.3}$$

$$g = n(t)I^\varepsilon a(\gamma)h(\beta) \qquad g \in G. \tag{A.4}$$

Here the element n of the nilpotent subgroup and other matrices a, h, k are the ones defined in (2). The stationary point of the nilpotent subgroup is $\dot{y} = (1, 0, 1)$:

$$\dot{y}n(t) = \dot{y}. \tag{A.5}$$

The coset space $Y = G/N$ is equivalent to the cone Y given by $y = \dot{y}g$ which can be defined in two realizations as

$$y = \exp(\gamma_k)s_k \qquad s_k = \dot{y}k(\phi) = (1, \sin\phi, \cos\phi) \tag{A.6}$$

$$y = \exp(\gamma_h)s_h \qquad s_h = \dot{y}h(\beta) = (\cosh\beta, \sinh\beta, (-1)^\varepsilon). \tag{A.7}$$

The connection between the above realizations is

$$\exp(\gamma_h)s_h = \exp(\gamma_k)s_k \tag{A.8}$$

or

$$\cosh\beta\exp(\gamma_h) = \exp(\gamma_k) \qquad \cos\phi = \frac{(-1)^\varepsilon}{\cosh\beta} \qquad \sin\phi = \tanh\beta. \tag{A.9}$$

Using (A.2) we get

$$F(y) = \exp(\gamma_h\sigma)F(s_h) = \exp(\gamma_k\sigma)F(s_k). \tag{A.10}$$

We know that the principal series of the irreducible representation in the compact basis (the group decomposition is $g = kak$) is unitary with respect to the scalar product

$$(F_1, F_2) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, F(s_k)\overline{F(s_k)} \tag{A.11}$$

if $\sigma = -\frac{1}{2} + \mathrm{i}\rho,\ \rho \in (0,\infty)$ [11]. Using the relation (A.10) we can introduce the scalar product in the space of representation in the mixed basis:

$$\langle F_1, F_2\rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \overline{\exp((\gamma_h - \gamma_k)\sigma))F(s_h)}F(s_k). \tag{A.12}$$

The invariant bilinear Hermitian form (A.12) can be written as

$$\langle F_1, F_2\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\beta\ \cosh\beta^{\sigma}\,\overline{F(s_h)}F(s_k). \tag{A.13}$$

From (A.1) we get the representation formulae corresponding to the above-mentioned realizations:

$$T^{\sigma}(g)F(s_k) = \exp((\gamma_k^g - \gamma_k)\sigma)F(s_{k^g}) \qquad g \in G \tag{A.14}$$

$$T^{\sigma}(g)F(s_h) = \exp((\gamma_h^g - \gamma_h)\sigma)F(s_{h^g}) \qquad g \in G. \tag{A.15}$$

Here s_{k^g} and s_{h^g} are defined as

$$\begin{aligned} \exp(\gamma_k^g)s_{k^g} &= \exp(\gamma_k)s_k g \\ \exp(\gamma_h^g)s_{h^g} &= \exp(\gamma_h)s_h g. \end{aligned} \tag{A.16}$$

We see that the natural representations for the maximal compact $k = SO(2)$ and non-compact $h = SO(1,1)$ subgroups corresponding to the realizations of the representations (A.14) and (A.15) are

$$T^{\sigma}(k(\phi_0))F(s_{k(\phi)}) = F(s_{k(\phi+\phi_0)}) \tag{A.17}$$

and

$$T^{\sigma}(h(\beta_0)F(s_{h(\beta)}) = F(s_{h(\beta+\beta_0)}). \tag{A.18}$$

By the help of the expansion formulae

$$\begin{aligned} F(s_k) &= \sum_{n=-\infty}^{\infty} C_n \exp(\mathrm{i}n\phi) \\ F(s_h) &= \int_{-\infty}^{\infty} \mathrm{d}\mu\, C_\mu \exp(\mathrm{i}\mu\beta) \end{aligned} \tag{A.19}$$

we can rewrite (A.17) and (A.18) as

$$\begin{aligned} T(k(\phi_0))\exp(\mathrm{i}n\phi) &= \exp(\mathrm{i}n\phi_0)\exp(\mathrm{i}n\phi) \\ T(h(\beta_0)\exp(\mathrm{i}\mu\beta) &= \exp(\mathrm{i}\mu\beta_0)\exp(\mathrm{i}\mu\beta) \end{aligned} \tag{A.20}$$

which coincide with the unitary irreducible representations of the subgroups $SO(2)$ and $SO(1,1)$.

Now we are ready to construct the unitary irreducible representation for the group $SO(1,2)$ in the mixed basis. Let us introduce the function $D(g)$:

$$D(g) = \langle F_1|T^{\sigma}(g)|F_2\rangle \tag{A.21}$$

in terms of the Hermitian bilinear form given by (A.13). Using the expansion formulae (A.19) we obtain

$$D(g) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\mu\, C_n \overline{C_\mu} d^{\sigma}_{\mu n}(g) \tag{A.22}$$

where $d^{\sigma}_{\mu n}(g)$ are the matrix elements of the unitary irreducible representation

$$d^{\sigma}_{\mu n}(g) = \langle\mu|T^{\sigma}(g)|n\rangle \qquad g = hak \in G. \tag{A.23}$$

By the help of the group property $T^\sigma(hak) = T(h)T^\sigma(a)T(k)$ and the expressions in (A.20) we obtain

$$d^\sigma_{\mu n}(g) = \exp(-\mathrm{i}\mu\beta)\langle\mu|T^\sigma(a)|n\rangle \exp(\mathrm{i}n\phi). \tag{A.24}$$

The integral representation for the matrix elements corresponding to the subgroup $a(\alpha)$ is (in the case $n = 0$)

$$d^\sigma_{\mu 0_k}(a) = \sum_{\varepsilon=0}^{1}\int_{-\infty}^{\infty} \mathrm{d}\beta\ \exp(-\mathrm{i}\mu\beta)(\cosh\beta\cosh\alpha + (-1)^\varepsilon \sinh\alpha)^\sigma. \tag{A.25}$$

Evaluating this integral we get

$$d^\sigma_{\mu 0_k}(a(\alpha)) = \frac{\cosh^{1/2}(\pi\rho)}{\cosh(\pi\rho) - \mathrm{i}\sinh(\pi\mu)}\,(P^{i\mu}_\sigma(\mathrm{i}\sinh\alpha) + P^{i\mu}_\sigma(-\mathrm{i}\sinh\alpha)). \tag{A.26}$$

Appendix B. The orthogonality condition

Consider the expression

$$B^{\sigma\sigma'}_{\mu\mu'} = \int_G \mathrm{d}g\, d^\sigma_{\mu 0_k}(g)\overline{d^{\sigma'}_{\mu' 0_k}(g)} \tag{B.1}$$

with $g = hak$. We first change the variables in the above integral:

$$hak = k'a'k'' \tag{B.2}$$

which is equivalent to passing from $g = hak$ to the Cartan decomposition $g' = k'a'k''$ [7]. Using the completness condition for the matrix elements of the maximal compact subgroup $K = SO(2)$:

$$\sum_{n=-\infty}^{\infty} |n\rangle\langle n| = 1 \tag{B.3}$$

and the equality

$$d^\sigma_{\mu 0_k}(g) = \langle\mu|T^\sigma(g)|0_k\rangle = \sum_{n=-\infty}^{\infty}\langle\mu|n\rangle\langle n|T^\sigma(g)|0_k\rangle = \sum_{n=-\infty}^{\infty}\langle\mu|n\rangle d^\sigma_{n0_k}(g) \tag{B.4}$$

we get

$$\begin{aligned} B^{\sigma\sigma'}_{\mu\mu'} &= \sum_{n,n'=-\infty}^{\infty}\langle\mu|n\rangle\langle\mu'|n'\rangle\int_{G'}\mathrm{d}g'\, d^\sigma_{n0_k}(g')\overline{d^{\sigma'}_{n'0_k}(g')} \\ &= \sum_{n,n'=-\infty}^{\infty}\langle\mu|n\rangle\overline{\langle\mu'|n'\rangle}\frac{\delta(\rho-\rho')\delta_{nn'}}{\rho\tanh\pi\rho} = \frac{\delta(\rho-\rho')\delta(\mu-\mu')}{\rho\tanh\pi\rho}. \end{aligned} \tag{B.5}$$

In (B.5) we used the orthogonality condition of the matrix elements in the Cartan basis [7, 9]. together with the orthogonality condition for the matrix elements of the maximal noncompact subgroup $SO(1, 1)$:

$$\langle\mu|\mu'\rangle = \delta(\mu-\mu'). \tag{B.6}$$

It is obviously equivalent to

$$\int_G \mathrm{d}g\, d^\sigma_{\mu 0_k}(g)\overline{d^{\sigma'}_{\mu' 0_k}(g)} = \frac{\delta(\rho-\rho')\delta(\mu-\mu')}{\rho\tanh\pi\rho} \tag{B.7}$$

where the invariant measure is $\mathrm{d}g = (\det_m g)^{1/2}\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\phi$.

Taking into account (14) we get the orthogonality condition for the wavefunctions:

$$\int_{-\infty}^{\infty}\mathrm{d}\alpha\ \Psi^\sigma_\mu\overline{\Psi^{\sigma'}_{\mu'}} = \frac{\delta(\mu-\mu')\delta(\sigma-\sigma')}{\rho\tanh\pi\rho}. \tag{B.8}$$

Appendix C. The addition theorem and the completeness condition

Note that the Legendre functions appear as the zonal spherical functions of the representation of the group $SO(1,2)$ if the elements of the group G has the Cartan decomposition $g = kak'$:

$$P^{\sigma}(\cosh\theta) = d^{\sigma}_{0_k,0_k}(a(\theta)). \tag{C.1}$$

Suppose that g_1 and g_2 are the elements of the group G which have the following decompositions:

$$g_j = h_j a_j k_j \qquad j = 1,2 \qquad \text{and} \qquad g_{12} = h_{12}a_{12}k_{12} = g_1^{-1}g_2. \tag{C.2}$$

Write the element g_{12} in the Cartan decomposition:

$$g_{12} = k'_{12}a'_{12}k_{12} \tag{C.3}$$

Using (C.1), (C.2) and (C.3) we obtain

$$\begin{aligned} P^{\sigma}(\cosh\theta_{12}) &= d^{\sigma}_{0_k,0_k}(a(\theta_{12})) = d^{\sigma}_{0_k,0_k}(k'_{12}a(\theta_{12})k_{12}) \\ &= d^{\sigma}_{0_k,0_k}(g_1^{-1}g_2) = \int_{-\infty}^{\infty} \mathrm{d}\mu\, d^{\sigma}_{0_k,\mu}(g_1^{-1}) d^{\sigma}_{\mu,0_k}(g_2). \end{aligned} \tag{C.4}$$

Making use of the property of the matrix elements

$$d^{-\sigma-1}_{0_k,\mu}(g^{-1}) = \overline{d^{\sigma}_{\mu,0_k}(g)} \tag{C.5}$$

and the equivalence of the representations T^{σ} and $T^{-\sigma-1}$ [9] we get the final result:

$$P^{\sigma}(\cosh\theta_{12}) = \int_{-\infty}^{\infty} \mathrm{d}\mu\, \overline{d^{\sigma}_{\mu,0_k}(g_1)} d^{\sigma}_{\mu,0_k}(g_2). \tag{C.6}$$

Here $\cosh\theta_{12}$ is defined from the algebraic equation (C.3) and is given by

$$\cosh\theta_{1,2} = \cosh\alpha_1 \cosh\alpha_2 \cosh(\beta_1 - \beta_2) - \sinh\alpha_1 \sinh\alpha_2 \tag{C.7}$$

which coincides with (22).

The completness condition for the matrix elements on the homogeneous space $g \in M$ are given by

$$\int_0^{\infty} \mathrm{d}\rho\, \rho \tanh \pi\rho \int_{-\infty}^{\infty} \mathrm{d}\mu\, d^{\sigma}_{\mu 0_k}(g)\overline{d^{\sigma}_{\mu 0_k}(g')} = \delta(g - g'). \tag{C.8}$$

To prove the above relation one considers the connection between the invariant differential operator (Laplace–Beltrami operator) on the manifold M with the Schrödinger equation. Since the Schrödinger equation has only a continuous specrum, the spectrum of the invariant differential operator should also be continuous. Therefore the discrete unitary series of the irreducible representation does not make a contribution and can be ignored. From the physical point of view the complementary series can also be ignored.

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